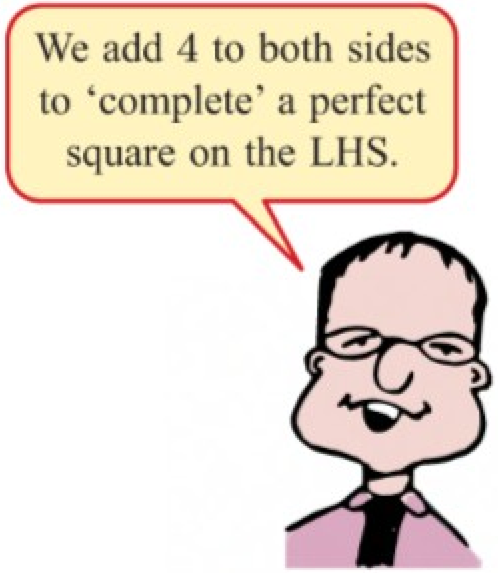
C COMPLETING THE SQUARE

Some quadratic equations, such as $x^2 + 4x - 7 = 0$, cannot be solved using the factorisation methods already practised. This is because the solutions are irrational.

Instead, we use a method called **completing the square**.

We can solve $x^2 + 4x - 7 = 0$ if we can rearrange it so there is a perfect square on the left hand side.

$$\begin{aligned} x^2 + 4x - 7 &= 0 \\ \therefore\ x^2 + 4x &= 7 \\ \therefore\ x^2 + 4x + 4 &= 7 + 4 \\ \therefore\ (x+2)^2 &= 11 \\ \therefore\ x + 2 &= \pm\sqrt{11} \\ \therefore\ x &= -2 \pm \sqrt{11} \end{aligned}$$


We add 4 to both sides to 'complete' a perfect square on the LHS.

The process of creating a **perfect square** on the left hand side is called **completing the square**.

The process of creating a **perfect square** on the left hand side is called **completing the square**.

From our previous study of perfect squares, we observe that:

$$
\begin{aligned}
&(x+3)^2 = x^2 + 2\times 3\times x + 3^2\\
&(x-5)^2 = x^2 + 2\times(-5)\times x + (-5)^2\\
\text{and in general that}\quad &(x+a)^2 = x^2 + 2ax + a^2.
\end{aligned}
$$

To complete a perfect square, the number we must add to both sides is found by **halving the coefficient of x, then squaring this value**.

Example 8

Self Tutor

For each of the following equations:

- **i** Find what must be added to both sides of the equation to create a perfect square on the LHS.
- **ii** Write the equation in the form $(x+p)^2 = k$.

a $x^2 + 8x = -5$

b $x^2 - 6x = 13$

Example 8

Self Tutor

For each of the following equations:

i Find what must be added to both sides of the equation to create a perfect square on the LHS.

ii Write the equation in the form $(x+p)^2 = k$.

a $x^2 + 8x = -5$ **b** $x^2 - 6x = 13$

a **i** In $x^2 + 8x = -5$, half the coefficient of x is $\frac{8}{2} = 4$.
So, we add 4^2 to both sides.

ii The equation becomes $x^2 + 8x + 4^2 = -5 + 4^2$

$$\therefore \quad (x+4)^2 = -5 + 16$$

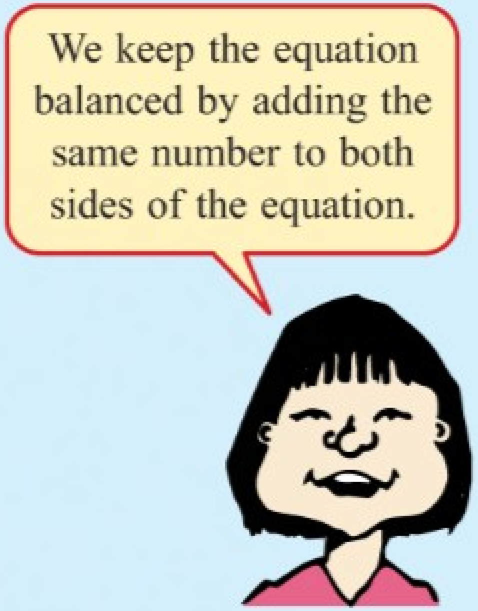
$$\therefore \quad (x+4)^2 = 11$$

b **i** In $x^2 - 6x = 13$, half the coefficient of x is $\frac{-6}{2} = -3$.
So, we add $(-3)^2 = 3^2$ to both sides.

ii The equation becomes $x^2 - 6x + 3^2 = 13 + 3^2$

$$\therefore \quad (x-3)^2 = 13 + 9$$

$$\therefore \quad (x-3)^2 = 22$$


We keep the equation balanced by adding the same number to both sides of the equation.

Example 9

Self Tutor

Solve for x by completing the square, leaving answers in simplest radical form:

a $x^2 + 2x - 2 = 0$

b $x^2 - 5x + 3 = 0$

Example 9 **Self Tutor**

Solve for x by completing the square, leaving answers in simplest radical form:

a $x^2 + 2x - 2 = 0$ **b** $x^2 - 5x + 3 = 0$

a
$$\begin{aligned} x^2 + 2x - 2 &= 0 \\ \therefore\ x^2 + 2x \quad &= 2 && \{\text{moving the constant term to the RHS}\} \\ \therefore\ x^2 + 2x + 1^2 &= 2 + 1^2 && \{\text{adding } (\tfrac{2}{2})^2 = 1^2 \text{ to both sides}\} \\ \therefore\ (x+1)^2 &= 3 \\ \therefore\ x + 1 &= \pm\sqrt{3} \\ \therefore\ x &= -1 \pm \sqrt{3} \end{aligned}$$

b
$$\begin{aligned} x^2 - 5x + 3 &= 0 \\ \therefore\ x^2 - 5x \quad &= -3 && \{\text{moving the constant term to the RHS}\} \\ \therefore\ x^2 - 5x + (\tfrac{5}{2})^2 &= -3 + (\tfrac{5}{2})^2 && \{\text{adding } (\tfrac{-5}{2})^2 = (\tfrac{5}{2})^2 \text{ to both sides}\} \\ \therefore\ (x - \tfrac{5}{2})^2 &= -3 + \tfrac{25}{4} \\ \therefore\ (x - \tfrac{5}{2})^2 &= \tfrac{13}{4} \\ \therefore\ x - \tfrac{5}{2} &= \pm\sqrt{\tfrac{13}{4}} \\ \therefore\ x &= \tfrac{5}{2} \pm \tfrac{\sqrt{13}}{2} \\ \therefore\ x &= \frac{5 \pm \sqrt{13}}{2} \end{aligned}$$

If $(x-a)^2 = k$ where $k > 0$, then $x = a \pm \sqrt{k}$.

1 For each of the following equations:

- **i** Find what must be added to both sides of the equation to create a perfect square on the LHS.
- **ii** Write each equation in the form $(x+p)^2 = k$.

a $x^2 + 2x = 5$

b $x^2 - 2x = -7$

c $x^2 + 6x = 2$

d $x^2 - 6x = -3$

e $x^2 + 10x = 1$

f $x^2 - 8x = 5$

g $x^2 + 12x = 13$

h $x^2 + 5x = -2$

i $x^2 - 7x = 4$

2 Solve for x by completing the square, leaving answers in simplest radical form:

a $x^2 - 4x + 1 = 0$

b $x^2 - 2x - 2 = 0$

c $x^2 - 4x - 3 = 0$

d $x^2 + 2x - 1 = 0$

e $x^2 + 4x + 1 = 0$

f $x^2 + 6x + 3 = 0$

g $x^2 + 3x + 2 = 0$

h $x^2 + 8x + 14 = 0$

i $x^2 - 3x - 1 = 0$

Example 10

Self Tutor

Solve for x by completing the square: $x^2 - 4x + 6 = 0$

Example 10

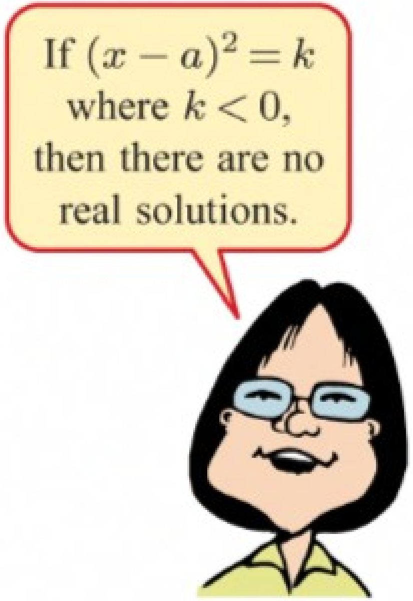
Self Tutor

Solve for x by completing the square: $x^2 - 4x + 6 = 0$

$$\begin{aligned}
& x^2 - 4x + 6 = 0 \\
\therefore\ & x^2 - 4x = -6 && \{\text{moving the constant term to the RHS}\} \\
\therefore\ & x^2 - 4x + 2^2 = -6 + 2^2 && \{\text{adding } (-\tfrac{4}{2})^2 = 2^2 \text{ to both sides}\} \\
\therefore\ & (x-2)^2 = -2
\end{aligned}$$

which is impossible as $(x-2)^2$ cannot be < 0.

$\therefore$ no real solutions exist.


If $(x-a)^2 = k$ where $k < 0$, then there are no real solutions.

3 If possible, solve for x by completing the square:

a $x^2 + 2x + 4 = 0$
b $x^2 - 5x + 6 = 0$
c $x^2 - 6x + 11 = 0$
d $x^2 + x - 1 = 0$
e $x^2 + 5x - 2 = 0$
f $x^2 - 7x + 13 = 0$

3 If possible, solve for x by completing the square:

a $x^2 + 2x + 4 = 0$
b $x^2 - 5x + 6 = 0$
c $x^2 - 6x + 11 = 0$
d $x^2 + x - 1 = 0$
e $x^2 + 5x - 2 = 0$
f $x^2 - 7x + 13 = 0$

Example 11

Self Tutor

Solve the equation $3x^2 + 6x - 2 = 0$ by completing the square.

Example 11 **Self Tutor**

Solve the equation $3x^2 + 6x - 2 = 0$ by completing the square.

$$\begin{aligned}
3x^2 + 6x - 2 &= 0 \\
\therefore\ x^2 + 2x - \tfrac{2}{3} &= 0 && \{\text{dividing both sides by } 3\} \\
\therefore\ x^2 + 2x \qquad &= \tfrac{2}{3} \\
\therefore\ x^2 + 2x + 1^2 &= \tfrac{2}{3} + 1^2 && \{\text{adding } (\tfrac{2}{2})^2 = 1^2 \text{ to both sides}\} \\
\therefore\ (x+1)^2 &= \tfrac{5}{3} \\
\therefore\ x + 1 &= \pm\sqrt{\tfrac{5}{3}} \\
\therefore\ x &= -1 \pm \tfrac{\sqrt{5}}{\sqrt{3}} \times \tfrac{\sqrt{3}}{\sqrt{3}} && \{\text{writing the RHS with integer denominator}\} \\
\therefore\ x &= \frac{-3 \pm \sqrt{15}}{3}
\end{aligned}$$

4 Solve by completing the square:

- **a** $2x^2 + 4x - 1 = 0$
- **b** $3x^2 - 12x + 7 = 0$
- **c** $5x^2 - 10x + 3 = 0$
- **d** $2x^2 + 12x - 5 = 0$
- **e** $2x^2 - 2x + 3 = 0$
- **f** $3x^2 + 3x - 1 = 0$

5
- **a** Solve by completing the square: $x^2 + bx + c = 0$
- **b** Under what conditions does the equation $x^2 + bx + c = 0$ have:
  - **i** two real solutions
  - **ii** one real solution
  - **iii** no real solutions?

1 **a** **i** 1^2 **ii** $(x+1)^2 = 6$
b **i** 1^2 **ii** $(x-1)^2 = -6$
c **i** 3^2 **ii** $(x+3)^2 = 11$
d **i** 3^2 **ii** $(x-3)^2 = 6$
e **i** 5^2 **ii** $(x+5)^2 = 26$
f **i** 4^2 **ii** $(x-4)^2 = 21$
g **i** 6^2 **ii** $(x+6)^2 = 49$
h **i** $(\frac{5}{2})^2$ **ii** $(x+\frac{5}{2})^2 = \frac{17}{4}$
i **i** $(\frac{7}{2})^2$ **ii** $(x-\frac{7}{2})^2 = \frac{65}{4}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = 1 \pm \sqrt{3}$ **c** $x = 2 \pm \sqrt{7}$
d $x = -1 \pm \sqrt{2}$ **e** $x = -2 \pm \sqrt{3}$ **f** $x = -3 \pm \sqrt{6}$
g $x = -1$ or -2 **h** $x = -4 \pm \sqrt{2}$ **i** $x = \dfrac{3 \pm \sqrt{13}}{2}$

3 **a** no real solution **b** $x = 3$ or 2 **c** no real solution
d $x = \dfrac{-1 \pm \sqrt{5}}{2}$ **e** $x = \dfrac{-5 \pm \sqrt{33}}{2}$ **f** no real solution

4 **a** $x = \dfrac{-2 \pm \sqrt{6}}{2}$ **b** $x = \dfrac{6 \pm \sqrt{15}}{3}$ **c** $x = \dfrac{5 \pm \sqrt{10}}{5}$
d $x = \dfrac{-6 \pm \sqrt{46}}{2}$ **e** no real solution **f** $x = \dfrac{-3 \pm \sqrt{21}}{6}$

5 **a** $x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$
b **i** $b^2 - 4c > 0$ **ii** $b^2 - 4c = 0$ **iii** $b^2 - 4c < 0$